# Introduction to Mathematical Modeling

## Mathematical models

## An overview of the book

## Some modeling approaches

## Modeling for decision making

# Compartmental Models

## Introduction

## Exponential decay and radioactivity

## Case study: detecting art forgeries

## Case study: Pacific rats colonize New Zealand

## Lake pollution models

## Case study: Lake Burley Griffin

## Drug assimilation into the blood

Summary of skills developed here:

Be able to model a sequence of processes using the compartmental technique.

Understand how to find numerical solutions, with Maple or Maple, and analytically each of the sequence of compartments.

Sketch graphs of solutions in each compartment over time.

Establish how changes parameters impact on these solutions.

## Case study: dull, dizzy, or dead?

## Cascades of compartments

## First-order linear Des

Summary of skills developed here

Identify a linear differential equation for which there exists a unique solution.

Understand the significance of the two separate parts to the solution.

## Equilibrium points and stability

## Case study: money, money, money makes the world go around

# Models of Single Populations

## Exponential growth

Summary of skills developed here

Formulate differential equations for single population, including immigration, emigration or harvesting.

Obtain exact solution by solving the differential equation.

For simple solution draw general sketches.

## Density-dependent growth

## Limited growth with harvesting

## Case study: anchovy wipe-out

## Case study: how can 2 × 106 birds mean rare?

## Discrete population growth and chaos

Oscillatory growth

When the intrinsic reproduction rate is increased further so that r > 1 it becomes possible for the population to increase above the carrying capacity, but in the iteration it then falls below K.

A damped oscillation results, where the oscillations become smaller with time, as seen in Figure 3.10.

As r increases the amplitude of these damped oscillations increases.

Once the population is above the carrying capacity K the reproduction rate r(Xn) now is negative (corresponding to the death rate being higher than the birth rate).

Thus the next breeding season on the population drops below the carrying capacity.

In the continuous model the reproduction rate changed instantaneously; with the discrete model, however, there is a delay of one breeding season before the reproduction rate can adjust to the change in population.

This is why it is possible for the population to jump above the carrying capacity in the discrete model, which is not the in the continuous model.

As we increase r further (see Figure 3.10), for r = 2.2 the population is again oscillating about the carrying capacity, but the amplitude of the oscillations appears to be constant.

We call this oscillation a 2-cycle since the population size is repeated every second breeding cycle once the initial transients have died out.

As r increase further, the amplitude of the oscillation increases.

It is possible to prove, using the difference equation (3.19), that stable 2-cycles persist if 2 r < 2.4.

This illustrates an increasingly common approach in mathematics, where computer experiments suggest results which are then proved using analysis.

Somewhere between r = 2.4 and r = 2.5 the 2-cydes become unstable.

Period doubling and chotic growth

Again, the trend here is for the amplitude of the 2-cycles to increase but they do not increase indefinitely.

We see from Figure 3.11 that the 2-cycle has become a 4-cycle by r = 2.5, where the values repeat themselves every 4 breeding cycles.

As r increases further this becomes an 8-cycle and then a 16-cycle and so on.

When we try 2.6 some entirely new behavior is observed.

The population does not grow in ordered cycles but seems to change randomly in each breeding period.

We call this type of growth chaotic, and it is illustrated in Figure 3.11 for r = 2.7.

Another feature of chaotic growth is that it is very sensitive to a change in the initial population.

This feature of chaotic growth is illustrated in Figure 3.12 with two initial population sizes of 100 and 101.

Although the difference between the populations is only 1 initially, after some time they become very different.

Both populations increasing and decreasing in a random-like pattern but there are significant differences in the graphs.

If we try to plot a similar graph, using initial conditions Xo = 1000 Xo = 1001 for any value of r which gave non-chaotic growth (r < 2.6) , the graphs of the two populations become indistinguishable after only a few breeding seasons; see exercises, Question 10.

The discovery that simple population models like this can give rise to chaotic growth had an extremely profound effect.

Previously, when presented with records of population growth which seemed to have the population growing in a random manner, it thought that this due to some external factor such as climate, environment, etc.

Now, however, it is known that the random-like behavior may be a natural feature of the way that the population grows, that is, a nonlinear response to a time delay in compensating the reproduction rate to count for crowding.

(It should be noted here that thus that large values of r are only valid for certain populations.)

## Time-delayed regulation

## Case study: Australian blowflies

# Numerical Solution of Differential Equations

## Introduction

## Basic numerical schemes

## Computer implementation using Maple and MATLAB

## Instability

## Discussion

Summary of skills developed here:

Understand the general principles of applying numerical methods, and how the process contributes towards generating errors computed solutions.

Understand the notion of a stable or unstable numerical scheme.

Understand the difference between an exact and an estimated solution.

Know how to choose a numerical scheme offered by Maple.

## Exercises for Chapter 4

# Interacting Population Models

## Introduction

## An epidemic model for influenza

## Predators and prey

## Case study: Nile Perch catastrophe

## Competing species

## Case study: aggressive protection of lerps and nymphs

## Model of a battle

## Case study: rise and fall of civilizations

## Exercise

# Phase-Plane Analysis

## Introduction

## Phase-plane analysis of epidemic model

## Analysis of a battle model

## Analysis of a predator-prey model

## Analysis of competing species models

## The predator-prey model revisited

## Case study: bacteria battle in the gut

## Exercise

# Linearization Analysis

## Introduction

## Linear theory

## Applications of linear theory

## Nonlinear theory

## Applications of nonlinear theory

## Exercise

# Some Extended Population Models

## Introduction

## Case study: competition, predation, and diversity

## Extended predator-prey model

## Case study: lemming mass suicides?

## Case study: prickly pear meets its moth

## Case study: geese defy mathematical convention

## Case study: possums threaten New Zealand cows

## Exercise

# Formulating Basic Heat Models

## Introduction

## Some basic physical laws

## Model for a hot water heater

## Heat conduction and Fourier’s law

## Heat conduction through a wall

## Radial heat conduction

## Heat fins

## Exercise

# Solving Time-Dependent Heat Problems

## The cooling coffee problem revisited

## The water heater problem revisited

## Case study: it’s hot and stuffy in the attic

## Spontaneous combustion

## Case study: fish and chips explode

## Exercise

# Solving Heat Conduction Problems

## Boundary condition problems

## Heat loss through a wall

## Case study: double glazing: what’s it worth?

## Insulating a water pipe

## Cooling a computer chip

## Exercise

# Introduction to Partial Differential Equations

## The heat conduction equation

## Oscillating soil temperatures

## Case study: detecting land mines

## Lake pollution revisited

## Exercise

# Appendix A: Differential Equations

## Properties of differential equations

A differential equation is an equation which involve the derivative(s) of some unknown function.

Such equations occur in many forms and thus specific terminology, outlined be1ow, is used to classify them.

Solving a differential equation invo1ves finding an expression for the unknown function, for which there is a variety of analytical techniques availab1e.

Since not all equations be solved analytically, numerical methods have been developed are widely used.

Some analytical techniques relevant to this text introduced below, with a brief introduction to numerical methods covered in Chapter 4.

## Solution by inspection

Some differential equations sufficiently simple for us to spot the general solution by inspection.

This requires a good knowledge of the derivatives of the elementary functions such as polynomials, exponential functions, hyperbolic functions and trigonometric functions.

## First-order separable equations

## First-order linear equations

## Homogeneous equations

## Inhomogeneous equations

# Appendix B: Further Mathematics

## Linear algebra

## Partial derivatives and Taylor expansions

## Review of complex numbers

## Hyperbolic functions

## Integration using partial fractions

# Appendix C: Notes on Maple and MATLAB

## Brief introduction to Maple

## Using Maple to solve DEs

## Brief introduction to MATLAB

## Solving differential equations with MATLAB

# Appendix D: Units and Scaling

## Scaling differential equations

## SI Units

#